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THE EXISTENCE AND STABILITY OF THE EQUILIBRIA OF MECHANICAL SYSTEMS WITH CONSTRAINTS PRODUCED BY LARGE POTENTIAL FORCES†

A. A. BUROV

e-mail: aburov@ccas.ru

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A modification of Routh's theorem is investigated for systems with unilateral constraints produced by large potential forces, which enables steady motions to be found and the sufficient conditions for their stability to be investigated. The problem of an orbital "monkey bridge" is considered as an example. © 2003 Elsevier Science Ltd. All rights reserved.

As is well known, Routh's method [1, 2] and its modifications [3, 4] enable not only the problem of the existence of steady motions of mechanical systems with bilateral constraints or those possessing first integrals to be effectively solved, but also enables the sufficient conditions for their stability and instability to be investigated.

The sufficient conditions for the stability of the equilibria of a system subject to unilateral constraints, both with reactions which vanish and which do not vanish, were obtained in [5] and were subsequently generalized in [6] (see also [7]). An effective method of analysing these conditions within the framework of the above approach was proposed in [8]. The methods of investigating the stability of periodic motions developed for systems with unilateral constraints, and also the theory of the bifurcations of such motions [9], were illustrated in [9–11] using numerous examples.

It was shown in [12], that for a correct description of the dynamics of constrained mechanical systems, it is necessary in a number of cases to have information on the mechanical origin of the forces producing these constraints. Within the framework of the idea proposed by Caratheodory, systems with unilateral constraints can be regarded as systems acted upon by large potential forces. A justification of this hypotheses, started in the 1950s [13], was proposed in [12, 14] (see also [10], where systematic explanations are given). In particular, using such forces, well-known results on the stability of periodic motions of systems subject to a unilateral constraints, were proved. Numerous references to other publication on this subject can be found in the monographs [9–11, 15].

The problem of the orbital "monkey bridge", considered as an example, belongs to a large class of important practical problems on the motion of orbital tethered systems [16–12], for each of which a fundamental analysis is needed of the role which non-retaining constraints play in them. Certain features of the existence and stability of the steady motions of an orbital tethered system with a massive tether were investigated previously in [22].

1. THE FUNDAMENTAL IDEA OF ROUTH'S THEORY FOR SYSTEMS CONSTRAINED BY BILATERAL CONSTRAINTS

To investigate the steady motions of mechanical systems, constrained by a bilateral constraint

$$f(\mathbf{x}) = 0, \quad \mathbf{x} \in R^n \tag{1.1}$$

and acted upon by forces with a potential $U = U(\mathbf{x})$, within the framework of Routh's method, Routh's function

$$W(\mathbf{x}, \lambda) = U(\mathbf{x}) + \lambda f(\mathbf{x}) \tag{1.2}$$

is set up and from the equations (the subscript $\mathbf{x}$ denotes a partial derivative with respect to $\mathbf{x}$)

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$$W_{\mathbf{x}} = U_{\mathbf{x}} + \lambda f_{\mathbf{x}} = 0 \tag{1.3}$$

supplemented by Eq. (1.1), their critical points are determined, to which the steady motions correspond. If the solution of Eqs (1.3) is represented in the parametric form $\mathbf{x} = \mathbf{x}_0(\lambda)$, the value of the parameter λ_0 corresponding to it is found from the equation

$$f(\mathbf{x}_0(\lambda)) = 0 \tag{1.4}$$

An investigation of the sufficient conditions for the stability of the steady motions obtained in this way reduces to determining the type of these critical points by analysing the sign-definiteness of the restriction of the function U to the surface (1.1) in a small neighbourhood of these critical points. For non-degenerate critical points, the conditions of stability can be obtained by investigating the sign-definiteness of the restriction of the quadratic form

$$\delta^2 W = \delta^2 U + \lambda \delta^2 f = \tfrac{1}{2}((U_{\mathbf{xx}} + \lambda f_{\mathbf{xx}})\delta\mathbf{x}, \delta\mathbf{x}) \tag{1.5}$$

to the linear manifold

$$\delta I = \{\delta\mathbf{x}: (f_{\mathbf{x}}, \delta\mathbf{x}) = 0\} \tag{1.6}$$

Here and henceforth all the derivatives are evaluated on the steady motions $(\mathbf{x}_0, \lambda_0)$ investigated.

If the number of negative eigenvalues of the quadratic form obtained in this way is equal to zero, we say that the degree of instability is equal to zero. Then the steady motion is Lyapunov stable, or, using the terminology of celestial mechanics, we have secular stability. If the number of negative eigenvalues is odd, we have instability. If the number of negative eigenvalues is even and no less than two, Routh's theorem does not enable one to draw any conclusions regarding the stability of the Lyapunov solution, but gyroscopic stabilization is possible.

The stability of degenerate critical points is investigated using higher-order forms, which arise when the function W is expanded in series in powers of the perturbations $\delta\mathbf{x}$ in the neighbourhood of the critical point $(\mathbf{x}_0, \lambda_0)$. There are general theorems, going back to Poincaré, which enable one to investigate the stability problem in degenerate cases.

2. THE STEADY MOTIONS OF A CONSTRAINT-FREE SYSTEM

We will now assume that the system in question is constraint-free, but an additional force acts on it with a potential (compare with the case considered previously in [23])

$$U_N = \tfrac{1}{2} N\varphi(\mathbf{x}) \tag{2.1}$$

which depends on the positive parameter N, and either

$$\varphi(\mathbf{x}) = f^2(\mathbf{x}) \tag{2.2}$$

or

$$\varphi(\mathbf{x}) = \begin{cases} f^2(\mathbf{x}), & \mathbf{x} \in \mathscr{E}_+ \cup \mathscr{E} \\ 0, & \mathbf{x} \in \mathscr{E}_+ \end{cases} \tag{2.3}$$

The regions $\mathscr{E}_\pm$ and the surface $\mathscr{E}$ are defined by the relation

$$\mathscr{E}_- = \{\mathbf{x}: f(\mathbf{x}) < 0\}, \quad \mathscr{E}_+ = \{\mathbf{x}: f(\mathbf{x}) < 0\}, \quad \mathscr{E} = \{\mathbf{x}: f(\mathbf{x}) = 0\} \tag{2.4}$$

respectively. We will consider the general situation when the smooth surface $\mathscr{E}$ separates the $(n-1)$-dimensional regions $\mathscr{E}_-$ and $\mathscr{E}_+$.

We will assume that, for sufficiently large values of N, the force with potential (2.1), (2.2) produces the bilateral constraint (1.1), while the force with potential (2.1), (2.3) produces the unilateral constraint

$$f(\mathbf{x}) \leq 0, \quad \mathbf{x} \in R^n \tag{2.5}$$

The steady motions of the system will be found as the critical points of the potential

$$W(\mathbf{x}; N) = U(\mathbf{x}) + U_N(\mathbf{x}) \tag{2.6}$$

They are given by the relations

$$W_{\mathbf{x}}(\mathbf{x}; N) = U_{\mathbf{x}} g(\mathbf{x}) + Nf(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) = 0 \tag{2.7}$$

which hold both for a potential of the form (2.1), (2.2) and for a potential of the form (2.1), (2.3) in the regions $\mathscr{E} \cup \mathscr{E}_+$. For the potential (2.1), (2.3) in the region $\mathscr{E}_-$ we have the equations

$$W_{\mathbf{x}}(\mathbf{x}; N) = U_{\mathbf{x}} = 0$$

Suppose

$$\lambda = Nf(\mathbf{x})$$

Then system (2.7), consisting of n equations in the variables $\mathbf{x}$, is equivalent to the system

$$U_{\mathbf{x}} + \lambda f_{\mathbf{x}} = 0, \quad f(\mathbf{x}) = \varepsilon\lambda, \quad \varepsilon = N^{-1} \tag{2.8}$$

which consists of $n + 1$ equations in the $n + 1$ unknowns $(\mathbf{x}, \lambda)$. The first subsystem of (2.8) is identical with Eq. (1.3). The second subsystem of (2.8) converts into Eq. (1.1) of the surface $\mathscr{E}$ as $\varepsilon \mapsto 0$.

We will seek a solution of this system in the form of an expansion in series in powers of ε

$$\mathbf{x} = \mathbf{x}_0 + \varepsilon\mathbf{x}_1 + \ldots, \quad \lambda = \lambda_0 + \varepsilon\lambda_1 + \ldots$$

Substituting these expansions into Eqs (2.8) and equating terms of like powers of the parameter ε, we obtain, in the zeroth and first approximations

$$U_{\mathbf{x}}^0 + \lambda_0 f_{\mathbf{x}}^0 = 0, \quad f^0 = 0 \tag{2.9}$$

$$\begin{gathered} \mathbf{A}\mathbf{x}_1 + f_{\mathbf{x}}^0\lambda_1 = 0, \quad (f_{\mathbf{x}}^0, \mathbf{x}_1) = \lambda_0, \quad \mathbf{A} = U_{\mathbf{xx}}(\mathbf{x}_0) + \lambda_0 f_{\mathbf{xx}}(\mathbf{x}_0) \\ f^0 = f(\mathbf{x}_0), \quad f_{\mathbf{x}}^0 = f_{\mathbf{x}}(\mathbf{x}_0), \quad U_{\mathbf{x}}^0 = U_{\mathbf{x}}(\mathbf{x}_0) \end{gathered} \tag{2.10}$$

Equations (2.9) are identical with Eqs (1.3) and (1.1), which describe the steady motions in the case of a bilateral constraint.

Equations (2.10), when the determinant of the matrix $\mathbf{A}$ is non-zero and $f_x^0 \neq 0$, i.e. $\mathbf{x}_0$ is a non-singular point of the surface $\mathscr{E}$, enable the steady motions to be obtained up to the first approximation from the formulae

$$\mathbf{x}_1 = \lambda_0 (\mathbf{A}^{-1} f_{\mathbf{x}}^0, f_{\mathbf{x}}^0)^{-1} \mathbf{A}^{-1} f_{\mathbf{x}}^0, \quad \lambda_1 = -\lambda_0 (\mathbf{A}^{-1} f_{\mathbf{x}}^0, f_{\mathbf{x}}^0)^{-1} \tag{2.11}$$

Then, since

$$f(\mathbf{x}_0 + \varepsilon\mathbf{x}_1 + \ldots) = f^0 + \varepsilon(f_{\mathbf{x}}^0, \mathbf{x}_1) + \ldots = \varepsilon\lambda_0 + \ldots \tag{2.12}$$

and $\varepsilon > 0$, then for $\lambda_0 > 0$ the critical point is situated inside the region $\mathscr{E}_+$, i.e. when there is a unilateral constraint – in the region of the constrained motion. When $\lambda_0 < 0$ the critical point is situated inside the region $\mathscr{E}_-$, which cannot occur in the case of a unilateral constraint. The case $\lambda_0 = 0$ requires a consideration of higher approximations.

In other words, for systems with a unilateral constraint the steady motions for which the corresponding critical point is situated on the constraint boundary only occur in the case when the active forces, defined by the potential U, press the system to the boundary of the constraint, or, in any case, do not repel it.

Remark. Equations (2.10) can be represented in the form

$$\mathbf{A}_f \mathbf{y} = \mathbf{b}$$

$$\mathbf{A}_f = \begin{Vmatrix} \mathbf{A} & f_{\mathbf{x}}^0 \\ f_{\mathbf{x}}^{0T} & 0 \end{Vmatrix}, \quad \mathbf{y} = \begin{Vmatrix} \mathbf{x}_1 \\ \lambda_1 \end{Vmatrix}, \quad \mathbf{b} = \begin{Vmatrix} 0 \\ \lambda_0 \end{Vmatrix}$$

If $\lambda_0 \neq 0$, this system, which is linear in the unknown y, is non-homogeneous and allows of a unique solution if the determinant of the matrix $\mathbf{A}_f$ is non-zero. If $\lambda_0 = 0$, this system is homogeneous, and its solution is non-zero only if the determinant of the matrix $\mathbf{A}_f$ vanishes.

The matrix of the form $\mathbf{A}_f$ considered arises in the general theory of conditional extrema (see, for example, [4]).

3. THE SUFFICIENT CONDITIONS FOR STABILITY

To find the sufficient conditions for the stability of the steady motion considered as $N \mapsto \infty$, we will analyse the second variation of potential (2.6) on this steady motion. It has the form

$$\begin{aligned}2\delta^2 W &= ((U_{\mathbf{xx}} + Nf_{\mathbf{xx}} \otimes f_{\mathbf{xx}} + Nf(\mathbf{x})f_{\mathbf{xx}}) \cdot \delta\mathbf{x}, \delta\mathbf{x}) = \\ &= U_{x_i x_j}\delta x_i \delta x_j + Nf_{x_i} f_{x_j} \delta x_i \delta x_j + Nf(\mathbf{x}) f_{x_i x_j} \delta x_i \delta x_j\end{aligned} \tag{3.1}$$

We substitute the solution $\mathbf{x} = \mathbf{x}_0 + \varepsilon\mathbf{x}_1 + \ldots$ into this expression, expand its right-hand side in series in the small parameter ε and equate terms of like powers of ε. We obtain

$$2\delta^2 W = 2N\delta^2 W_0 + 2\delta^2 W_1 + \ldots$$

where

$$\delta^2 W_0 = (f_{\mathbf{x}}^0, \delta\mathbf{x})^2 \tag{3.2}$$

$$\delta^2 W_1 = (U_{\mathbf{xx}}^0 \delta\mathbf{x}, \delta\mathbf{x}) + (f_{\mathbf{x}}^0, \mathbf{x}_1) \cdot (f_{\mathbf{xx}}^0 \delta\mathbf{x}, \delta\mathbf{x}) + 2(f_{\mathbf{x}}^0, \delta\mathbf{x})(f_{\mathbf{xx}}^0 \mathbf{x}_1, \delta\mathbf{x}) \tag{3.3}$$

The lowest-order term in (3.2) is non-negative. It vanishes if the variation $\delta\mathbf{x}$ belong to a linear manifold

$$\delta I^0 = \{\delta\mathbf{x}: (f_{\mathbf{x}}^0, \delta\mathbf{x}) = 0\} \tag{3.4}$$

i.e. satisfaction of the sufficient condition for stability will be ensured if the restriction of the next-order term to this linear manifold is positive-definite.

The next-order term in (3.3) can now represented in the form

$$2\delta^2 W_1 = (U_{\mathbf{xx}}^0, \delta\mathbf{x}, \delta\mathbf{x}) + \lambda_0 (f_{\mathbf{xx}}^0, \delta\mathbf{x}, \delta\mathbf{x}) = (\mathbf{A}\delta\mathbf{x}, \delta\mathbf{x}) \tag{3.5}$$

Hence, the positive-definiteness of the restriction of the quadratic form (3.5) to linear manifold (3.4) ensures the positive-definiteness of the second variation $\delta^2 W$ for sufficiently small values of the parameter ε.

The propositions proved above of the existence and stability of the steady motions of a system with a unilateral constraint can be formulated in the form of the following assertion.

Assertion. Suppose $(\mathbf{x}_0, \lambda_0)$ is a non-singular critical point of Routh's function (1.2), such that

$$f^0 = 0, \quad f_{\mathbf{x}}^0 \neq 0, \quad \lambda_0 > 0$$

Then, for unilateral constraint (2.5), which can be achieved using large potential forces, a steady motion exists, the zeroth approximation of which is determined by this critical point.

If the restriction of the quadratic form

$$2\delta^2 W = (W_{\mathbf{xx}}^0 \delta\mathbf{x}, \delta\mathbf{x}) \tag{3.6}$$

to linear manifold (3.4) is positive-definite, the steady motion is Lyapunov stable for sufficiently large values of N.

Roughly speaking, the mechanical meaning of the stability conditions is that if the active force "clamps" the system to the constraint, instability can only develop in directions situated in the tangential plane to its boundary.

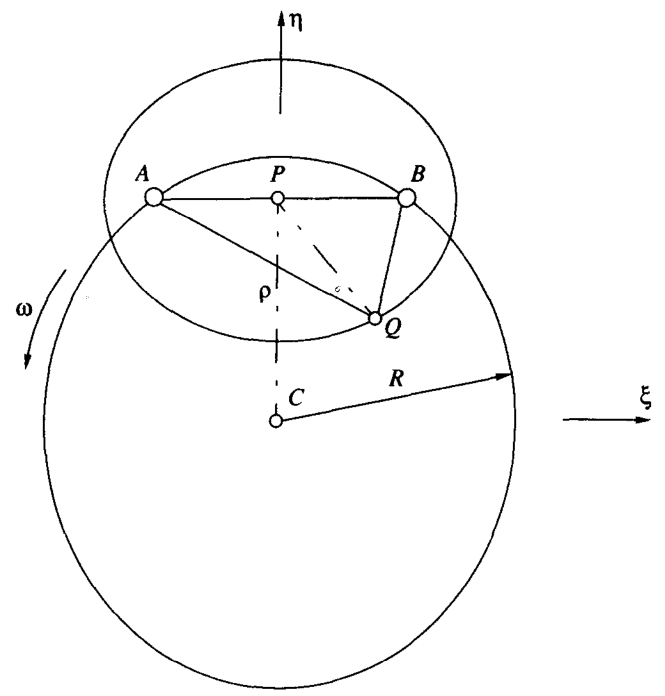

Fig. 1

4. THE PROBLEM OF A SPACE "MONKEY BRIDGE"

Consider the motion of a bead (a point mass) Q of mass m in a Newtonian attraction field with centre of attraction at the point C. We will assume that the bead Q moves in the same plane as a pair of satellites A and B, which move in succession in a common circular Kepler orbit of radius R. We will denote their orbital angular velocity by ω and the fixed distance between them by $2l$. The distance between the middle point P of the section AB and the centre of attraction C also does not change, and we will denote it by ρ, so that $l^2 + \rho^2 = R^2$ (Fig. 1).

We will assume that the bead Q can slide without friction along a weightless inextensible tether of length $2a$, which connects satellites A and B. Assuming that the bead Q does not leave orbital plane, we can conclude that it is situated inside an ellipse with foci at the points A and B. The semi-major axis of this ellipse is equal to a, and the semi-minor axis $b = (a^2 - l^2)^{1/2}$.

The equations of the motion. We will introduce a moving system of coordinates $C\xi\eta$, which rotates uniformly with orbital angular velocity ω around the perpendicular to the orbital plane passing through the point C. We will assume that the $C\xi$ axis is parallel to the section AB. The $C\eta$ axis is then perpendicular to the section AB and passes through its middle point P. Hence, the equation of the ellipse can be written in the form

$$f(\xi, \eta) = \xi^2/a^2 + (\eta - \rho)^2/b^2 - 1 = 0 \tag{4.1}$$

Its position is shown in Fig. 1.

We will denote the coordinates of the point Q in the orbital system of coordinates introduced above by (ξ, η). Then, the transformed potential of the system can be represented in the form

$$U_a = -m\omega^2[r^2/2 + R^3/r], \quad r = (\xi^2 + \eta^2)^{1/2}$$

In order to use the modified Routh's method proposed above, we will write Routh's function in the form

$$W = U_a + \chi f(\xi, \eta)/2$$

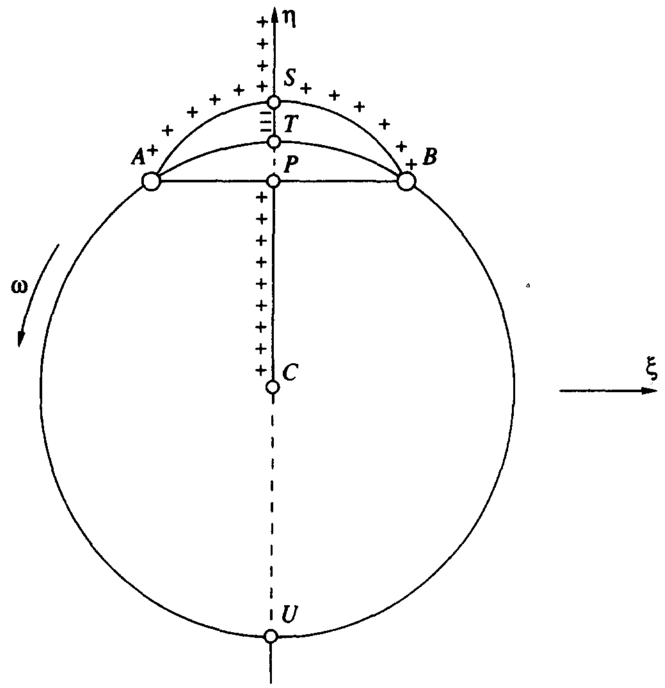

Fig. 2

The relative equilibria coincide with the critical points of this function. They can be obtained from the equations

$$\frac{\partial W}{\partial \xi} = \xi\Lambda_a = 0, \quad \frac{\partial W}{\partial \eta} = \eta\Lambda_b - \frac{\chi\rho}{b^2} = 0, \quad \frac{\partial W}{\partial \chi} = 0$$
$$\Lambda_c = -m\omega^2(1 - R^3/r^3) + \chi/c^2, \quad c \in \{a, b\} \tag{4.2}$$

We draw attention to the sign of the factor χ. If this factor is positive, the tether is stretched. If this factor is equal to zero, but relation (4.1) is satisfied, then, although the system is also under a constraint, the reaction of the constraint is equal to zero. In the remaining case the tether is not stretched, and the bead Q is not restrained.

From the first equation of (4.2) we will first obtain the particular solution for which $\xi = 0$. Then, by virtue of the last equation of (4.2) we obtain $\eta = \rho + \varepsilon b$, $\varepsilon = \pm 1$, where the factor χ can be found from the second equation of (4.2). it has the form

$$\chi = m\omega^2(1 - R^3/|\rho + \varepsilon b|^3)\varepsilon b$$

Suppose $\varepsilon = -1$. Then the tether is stretched if $b < \rho$, i.e. the point Q is situated between the points P and C (Fig. 2). For $\varepsilon = 1$ the factor χ is only positive when the point Q is situated outside the circular orbit of the points A and B. These relative equilibria are called vertical equilibria.

We will now assume that $\Lambda_a = 0$, i.e. the second factor on the left-hand side of the first equation of (4.2) vanishes. Then, expressing the quantity χ from this equation and substituting its value into the second relation of (4.2), we obtain

$$\chi(\eta[1/b^2 - 1/a^2] - \rho/b^2) = 0 \tag{4.3}$$

Equation (4.3) becomes a true numerical equality when $\chi = 0$, i.e. the point Q is in the same circular orbit as the points A and B. These relative equilibria are not isolated.

Equation (4.3) also has the solution

$$\eta = \rho a^2/(a^2 - b^2) = \rho a^2/l^2$$

From the equation of an ellipse we obtain that, on relative equilibria, corresponding to this solution

$$\eta = \varepsilon a[(1-\rho b/l^2)(1+\rho b/l^2)]^{1/2}$$

These equilibria (which we will call *inclined* equilibria) exist when the following condition is satisfied

$$a < lR/\rho$$

The set of vertical and inclined equilibria intersect if $\xi = 0$, i.e. if

$$a = lR/\rho$$

This bifurcation point, denoted by S, and the point P are conjugate to one another in the sense of an inversion transformation with respect to the circumference of the orbit.

For inclined equilibria

$$r^2 = \xi^2 + \eta^2 = a^2 + a^2\rho^2/l^2 = a^2R^2/l^2$$

Since $a > l$, we have $r > R$, and by virtue of relation (4.3) the tether is stretched on inclined equilibria. Stability of relative equilibria.

Stability of relative equilibria. Although one can use the variables (ξ, η) to investigate stability, it is more convenient to introduce the angular variable φ, such that

$$\xi = a\sin\varphi, \quad \eta = \rho + b\cos\varphi$$

assuming that the reaction of the constraint does not vanish. The transformed potential then has the form

$$U_a(\varphi) = -m\omega^2(\tfrac{1}{2}\Phi(\varphi) + R^3\Phi^{-1/2}(\varphi)), \quad \Phi(\varphi) = a^2\sin^2\varphi + (\rho + b\cos\varphi)^2$$

Then

$$\partial U_a/\partial\varphi = -m\omega^2\sin\varphi Q(\varphi)(1 - R^3/r^3), \quad Q(\varphi) = [a^2\cos\varphi - b(\rho + b\cos\varphi)]^{1/2}$$

This expression vanishes if the following equalities are satisfied

$$\sin\varphi = 0, \quad \cos\varphi = \varepsilon$$

i.e. on a pair of vertical equilibria, and also if

$$\cos\varphi = b\rho/(a^2 - b^2)$$

i.e. on a pair of inclined equilibria, symmetrical about the vertical, or if

$$r = R$$

i.e. on the family of non-isolated equilibria situated on a Kepler orbit.

To investigate the stability, we find the second derivative of the transformed potential

$$\partial^2 U_a/\partial\varphi^2 = -m\omega^2[\cos\varphi Q(\varphi) - \sin^2\varphi(a^2 - b^2)](1 - R^3/r^3)$$

For vertical relative equilibria the sufficient conditions for stability have the form

$$-m\omega^2\varepsilon(\varepsilon(a^2 - b^2) - b\rho)(1 - R^3/r^3) > 0$$

This indicates that when $\varepsilon = -1$ the relative equilibrium is stable if the point Q is situated between the point P and the attracting centre. If $\varepsilon = 1$, the relative equilibria are stable if the point Q is situated above the bifurcation point S. The relative equilibria belonging to the section ST are always unstable.

For inclined equilibria the sufficient condition for stability has the form

$$m\omega^2(a^2 - b^2)\sin^2\varphi[1 - R^3/r^3] > 0$$

This condition is satisfied for inclined relative equilibria, since on them the point Q is situated outside the orbit of the points A and B.

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REFERENCES

1. ROUTH, E. J., *A Treatise on the Stability of a Given State of Motion*. Macmillan, London, 1877.
2. ROUTH, E. J., *The Advanced Part of a Treatise on the Dynamics of a System of Rigid Bodies*. Macmillan, London, 1884.
3. SALVADORI, L., Un'osservazione su di un criterio di stabilita del, *Routh. Rend. Accad. Sci. fis. e math. Soc. naz. lett. ed arti. Napoli*, 1953, **20**, 1–2, 269–272.
4. RUBANOVSKII, V. N. and STEPANOV, S. Ya., Routh's theorem and Chetayev's method of constructing Lyapunov's function from integrals of the equations of motion. *Prikl. Mat. Mekh*., 1969, **35**, 5, 904–912.
5. IVANOV, A. P., Stability in systems with non-retaining constraints. *Prikl. Mat. Mekh*., 1984, **48**, 5, 725–732.
6. IVANOV, A. P., *The Modelling of Systems with Mechanical Collisions*. Izd. Mosk. Inst. Priborostroyeniya, Moscow, 1992.
7. VINNER, G. M., The stability of the equilibrium of a mechanical system with a non-retaining constraint. *Vestnik MGU. Ser. 1. Mat. Mekh*., 1989, 4, 54–57.
8. RAPOPORT, L. B., The stability of the equilibrium of systems with non-retaining constraints and sign-definiteness of a pencil of quadratic forms in a cone. *Prikl. Mat. Mekh*., 1992, **56**, 4, 597–603.
9. IVANOV, A. P., *The Dynamics of Systems with Mechanical Collisions*. International Education Programme, Moscow, 1997.
10. KOZLOV, V. V. and TRESHCHEV, D. V., *Billiards. A Genetic Introduction to the Dynamics of Systems with Collisons*. Izd. MGU, Moscow, 1991.
11. BELETSKII, V. V., *Reguläre und chaotische Bewegung starrer Körper*. Teubner, Stuttgart, 1995.
12. KOZLOV, V. V., The realization of non-integrable constraints in classical mechanics. *Dokl. Akad. Nauk SSSR*, 1983, **272**, 3, 550–554.
13. RUBIN, H. and UNGAR, P., Motions under a strong constraining force. *Communs Pure and Appl. Math*., 1957, **10**, 1, 65–87.
14. KOZLOV, V. V., A constructive method of substantiating the theory of systems with non-retaining constraints. *Prikl. Mat. Mekh*., 1988, **52**, 6, 883–894.
15. ZHURAVLEV, V. F. and FUFAYEV, N. A., *The Mechanics of Systems with Non-retaining Constraints*. Nauka, Moscow, 1993.
16. BANERJEE, A. K. and KANE, T. R., Pointing control, with tethers as actuators, of a space station supported platform. *J. Guid., Control and Dynamics*, 1993, **16**, 2, 396–399.
17. MOCCIA, A., VETRELLA, S. and GROSSI, M., Attitude dynamics and control of a vertical intergeometric radar tethered altimeter. *J. Guid., Control and Dynamics*, 1993, **16**, 2, 264–269.
18. LORENZINI, E. C., Three-mass tethered system for micro-/variable-g applications. *J. Guid., Control and Dynamics*, 1987, **10**, 3, 242–249.
19. MISRA, A. K., AMIER, Z. and MODI, V. J., Attitude dynamics of three body tethered systems. *Acta Astronaut*., 1988, **17**, 10, 1059–1068.
20. MISRA, A. K. and DIAMOND, G. S., Dynamics of a substellite system supported by two tethers. *J. Guid., Control and Dynamics*, 1986, **9**, 1, 12–16.
21. BELETSKY, V. V. and LEVIN, E. M., *Dynamics of Space Tether System*. Am. Astronaut. Soc., San Diego, 1993.
22. KRUPA, M., SCHAGERL, M., STEINDL, A., SZMOLYAN, P. and TROGER, H., Relative equilibria of tethered satellite systems and their stability for very stiff tethers. *Dynam. Syst*., 2001, **16**, 3, 253–278.
23. DERYABIN, M. V. and KOZLOV, V. V., The theory of systems with unilateral constraints. *Prikl. Mat. Mekh*., 1995, **59**, 4, 531–539.

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